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MATH4060 Complex Analysis
Homework 5 Suggested Solutions
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1. (Exercise 15 of Chapter 8 of [SS03]) Here are two properties enjoyed by automorphisms of the upper half-plane.

   (a) Suppose Φ is an automorphism of $\mathbb{H}$ that fixes three distinct points on the real axis. Then Φ is the identity.

   (b) Suppose (x_1, x_2, x_3) and (y_1, y_2, y_3) are two pairs of three distinct points on the real axis with
   $$x_1 < x_2 < x_3 \quad \text{and} \quad y_1 < y_2 < y_3.$$
   Prove that there exists (a unique) automorphism Φ of $\mathbb{H}$ so that $\Phi(x_j) = y_j$, $j = 1, 2, 3$. The same conclusion holds if $y_3 < y_1 < y_2$ or $y_2 < y_3 < y_1$.

Solution. (a) By Theorem 2.4 of Chapter 8 of [SS03], Φ is of the form

$$\Phi(z) = \frac{az+b}{cz+d}, \quad a, b, c, d \in \mathbb{R} \text{ and } ad - bc = 1.$$

Let $x_1, x_2, x_3 \in \mathbb{R}$ be the distinct fixed points of Φ, then

$$\Phi(x_i) = x_i \Rightarrow cx_i^2 + (d-a)x_i - b = 0, \quad \text{for } i = 1, 2, 3.$$

However, by the fundamental theorem of algebra, the equation above has at most two solutions. Hence, the equation above can only be the trivial equation $(0 = 0)$ which implies $c = d - a = -b = 0$. In particular, since $ad = 1$, this forces $a = d = \pm 1$. Plugging these values for a, b, c, d back into the form of Φ above, we find that
$$\Phi(z) = \frac{\pm z}{\pm 1} = z$$
the identity.

(b) We can define Φ implicitly by the equation of cross ratios

$$\frac{(z-x_1)(x_2-x_3)}{(z-x_3)(x_2-x_1)} = \frac{(\Phi(z)-y_1)(y_2-y_3)}{(\Phi(z)-y_3)(y_2-y_1)}$$

which by construction makes Φ map x_i to y_i for $i = 1, 2, 3$ respectively. The way to see this is that the equation above becomes

$$(\Phi(z)-y_3)(y_2-y_1)(z-x_1)(x_2-x_3) = (\Phi(z)-y_1)(y_2-y_3)(z-x_3)(x_2-x_1).$$

In the equation above, if $\Phi(z) = y_1$, then the right-hand side is 0 forcing $z = x_1$ on the left-hand side. Similarly, if $\Phi(z) = y_3$, then the left-hand side is 0 which forces $z = x_3$ on the right-hand side. Finally, if $\Phi(z) = y_2$, then the equation above reduces to

$$(z - x_1)(x_2 - x_3) = (z - x_3)(x_2 - x_1) \Rightarrow z(x_1 - x_3) = x_2(x_1 - x_3) \Rightarrow z = x_2.$$

We now solve for $\Phi(z)$ explicitly. Let $\alpha = (y_2 - y_1)(x_2 - x_3)$ and $\beta = (y_2 - y_3)(x_2 - x_1)$. Then re-arranging the equation above, we obtain

$$\begin{aligned}\Phi(z) &= \frac{(z - x_1)y_3\alpha - (z - x_3)y_1\beta}{(z - x_1)\alpha - (z - x_3)\beta} \\ &= \frac{(y_3\alpha - y_1\beta)z + (y_1x_3\beta - y_3x_1\alpha)}{(\alpha - \beta)z + (x_3\beta - x_1\alpha)}\end{aligned}$$

which has determinant

$$\begin{aligned}&(y_3\alpha - y_1\beta)(x_3\beta - x_1\alpha) - (y_1x_3\beta - y_3x_1\alpha)(\alpha - \beta) \\ &\quad = \alpha\beta(x_3y_3 + x_1y_1 - x_3y_1 - x_1y_3) \\ &\quad = \alpha\beta(x_3 - x_1)(y_3 - y_1) \\ &\quad = (y_2 - y_1)(x_2 - x_3)(y_2 - y_3)(x_2 - x_1)(x_3 - x_1)(y_3 - y_1)\end{aligned}$$

which is positive by the assumptions $x_1 < x_2 < x_3, y_1 < y_2 < y_3$. Hence, we see that Φ is an automorphism of $\mathbb{H}$. It remains to show uniqueness. Suppose $\tilde{\Phi}$ is another automorphism of $\mathbb{H}$ which maps x_i to y_i for $i = 1, 2, 3$ respectively. Then we see that $\tilde{\Phi}^{-1} \circ \Phi$ and $\Phi^{-1} \circ \tilde{\Phi}$ both have three distinct fixed points which by part (a) above implies they are both the identity. From here it is easy to deduce that $\Phi = \tilde{\Phi}$.

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2. (Exercise 21 of Chapter 8 of [SS03]) We consider conformal mappings to triangles.

   (a) Show that
   $$\int_0^z \zeta^{-\beta_1}(1 - \zeta)^{-\beta_2} d\zeta,$$
   with $0 < \beta_1 < 1, 0 < \beta_2 < 1$, and $1 < \beta_1 + \beta_2 < 2$, maps $\mathbb{H}$ to a triangle whose vertices are the images of $0, 1$, and ∞, and with angles $\alpha_1\pi, \alpha_2\pi$, and $\alpha_3\pi$, where $\alpha_j + \beta_j = 1$ and $\beta_1 + \beta_2 + \beta_3 = 2$.

   (b) What happens when $\beta_1 + \beta_2 = 1$?

   (c) What happens when $0 < \beta_1 + \beta_2 < 1$?

   (d) In (a), the length of the side of the triangle opposite angle $\alpha_j\pi$ is
   $$\frac{\sin(\alpha_j\pi)}{\pi}\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3).$$

Solution. (a) Note the typo the integral above (with corrections in red) in the textbook. Letting $S(z)$ be the integral above, we see that

$$S(z) = \int_0^z \zeta^{-\beta_1}(1 - \zeta)^{-\beta_2} d\zeta = e^{i\beta_2\pi} \int_0^z \zeta^{-\beta_1}(\zeta - 1)^{-\beta_2} d\zeta$$

which by Proposition 4.1 of Chapter 8 of [SS03], maps $\mathbb{R} \cup \{\infty\}$ to the triangle with vertices at the images of $0, 1, \infty$ with the desired conclusions on the angles. To see that S is a conformal mapping of $\mathbb{H}$ to the triangular region with vertices as above, we use the uniqueness of Theorem 4.7 of Chapter 8 of [SS03]. Let F be the conformal mapping of $\mathbb{H}$ to the triangle with the desired vertices, that is, $F(0) = S(0), F(1) = S(1), F(\infty) = S(\infty)$ and the same interior angles as above. Then by Theorem 4.7, F takes the form

$$F(z) = C_1 \int_0^z \zeta^{-\beta_1} (\zeta - 1)^{-\beta_2} d\zeta + C_2.$$

It remains to determine the values of the constants C_1, C_2. Since $S(0) = 0$, $F(0) = 0$ as well and hence $C_2 = 0$. Then the equation $F(1) = S(1)$ implies $C_1 = e^{i\beta_2\pi}$ as well.

(b) By Proposition 4.1, when $\beta_1 + \beta_2 = 1$, the interior angle at the image of infinity is π and therefore the image of S is the unbounded region with a line segment and two parallel lines as the boundary.

(c) Similarly, when $\beta_1 + \beta_2 < 1$, the image of S is the unbounded region with line segment and two non-parallel lines as the boundary.

(d) The sides are the images of $(-\infty, 0), (0, 1), (1, \infty)$ respectively. Recall the formula for the Beta function shown in Homework 3

$$B(\alpha, \beta) = \int_0^1 (1 - t)^{\alpha - 1} t^{\beta - 1} dt = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}$$

Then we first compute

$$\begin{aligned} \int_0^1 (-t)^{-\beta_2} t^{-\beta_1} dt &= \frac{\Gamma(1 - \beta_2)\Gamma(1 - \beta_1)}{\Gamma(2 - \beta_2 - \beta_1)} \\ &= \frac{\Gamma(\alpha_2)\Gamma(\alpha_1)}{\Gamma(1 - \alpha_3)} \\ &= \frac{\sin(\alpha_3\pi)}{\pi} \Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3) \end{aligned}$$

where we also used the symmetry of the Γ function around the line $\mathrm{Re}(s) = \frac{1}{2}$ (Theorem 1.4 of Chapter 6 of [SS03]).

For $\int_1^\infty (1-t)^{-\beta_2} t^{-\beta_1} dt$ one obtains the corresponding formula using the change of variables $t = s^{-1}$ and for $\int_{-\infty}^0 (1-t)^{-\beta_2} t^{-\beta_1} dt$ one obtains the corresponding formula using the change of variables $t = 1 - s^{-1}$.

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3. (Exercise 22 of Chapter 8 of [SS03]) If P is a simply connected region bounded by a polygon whose vertices $a_1, \ldots, a_n$ and angles $\alpha_1\pi, \ldots, \alpha_n\pi$ and F is a conformal map of the disk $\mathbb{D}$ to P, then there exist complex numbers $B_1, \ldots, B_n$ on the unit circle, and constants c_1 and c_2 so that

$$F(z) = c_1 \int_1^z \frac{d\zeta}{(\zeta - B_1)^{\beta_1} \cdots (\zeta - B_n)^{\beta_n}} + c_2.$$

[Hint: This follows from the standard correspondence between $\mathbb{H}$ and $\mathbb{D}$ and an argument similar to that used in the proof of Theorem 4.7.]

Solution. Recall the conformal map $G : \mathbb{D} \to \mathbb{H}$

$$G(z) = i\frac{1-z}{1+z}$$

with inverse $G^{-1} : \mathbb{H} \to \mathbb{D}$

$$G^{-1}(w) = \frac{i-w}{i+w}.$$

Then if F is a conformal map from $\mathbb{D}$ to P as in the question, $F \circ G^{-1}$ is a conformal map from $\mathbb{H}$ to P. By Theorem 4.6 of Chapter 8, we therefore have that

$$(F \circ G^{-1})(w) = c_1 \int_1^w \frac{d\zeta}{(\zeta - A_1)^{\beta_1} \cdots (\zeta - A_n)^{\beta_n}} + c_2$$

where c_1, c_2 are constants in $\mathbb{C}$. Then composing with G on the right, we find that

$$\begin{aligned}
F(z) &= c_1 \int_1^z \frac{d\left(\frac{i-\zeta}{i+\zeta}\right)}{\left(\frac{i-\zeta}{i+\zeta} - A_1\right)^{\beta_1} \cdots \left(\frac{i-\zeta}{i+\zeta} - A_n\right)^{\beta_n}} + c_2 \\
&= c_1 \int_1^z \frac{\frac{-2i}{(i+\zeta)^2} d\zeta}{\left(\frac{i-\zeta}{i+\zeta} - A_1\right)^{\beta_1} \cdots \left(\frac{i-\zeta}{i+\zeta} - A_n\right)^{\beta_n}} + c_2 \\
&= -2ic_1 \int_1^z \frac{d\zeta}{(i - \zeta - A_1)^{\beta_1} \cdots (i - \zeta - A_n)^{\beta_n}} + c_2 \\
&= c_1' \int_1^z \frac{d\zeta}{(\zeta - B_1)^{\beta_1} \cdots (\zeta - B_n)^{\beta_n}} + c_2
\end{aligned}$$

for $c_1' = 2ic_1$ and $B_j = A_j - i$ for $j = 1, \dots, n$ and after making the change of variables $\zeta \mapsto -\zeta$. ◀

4. (Exercise 24(a)(b) of Chapter 8 of [SS03]) The elliptic integrals K and K' defined for $0 < k < 1$ by

$$K(k) = \int_0^1 \frac{dx}{((1-x^2)(1-k^2x^2))^{1/2}} \quad \text{and} \quad K'(k) = \int_1^{1/k} \frac{dx}{((x^2-1)(1-k^2x^2))^{1/2}}$$

satisfy various interesting identities. For instance:

(a) Show that if $\tilde{k}^2 = 1 - k^2$ and $0 < \tilde{k} < 1$, then

$$K'(k) = K(\tilde{k}).$$

[Hint: Change variables $x = (1 - \tilde{k}^2 y^2)^{-1/2}$ in the integral defining $K'(k)$.]

(b) Prove that if $\tilde{k}^2 = 1 - k^2$, and $0 < \tilde{k} < 1$, then

$$K(k) = \frac{2}{1+\tilde{k}} K\left(\frac{1-\tilde{k}}{1+\tilde{k}}\right).$$

[Hint: Change variables $x = 2t/(1 + \tilde{k} + (1 - \tilde{k})t^2)$.]

Solution. (a) Following the hint, we let $x = (1 - \tilde{k}^2 y^2)^{-1/2}$, then $dx = \tilde{k}^2 y(1 - \tilde{k}^2 y^2)^{-3/2} dy$ and we see that when $x = 1$, $y = 0$ and when $x = 1/k$, $y = 1$. Then plugging into $K'(k)$, we have

$$\begin{aligned} K'(k) &= \int_1^{1/k} \frac{dx}{((x^2-1)(1-k^2x^2))^{1/2}} \\ &= \int_0^1 \frac{\tilde{k}^2 y(1-\tilde{k}^2y^2)^{3/2} dy}{\left[\left((1-\tilde{k}^2y^2)^{-1} - 1\right)\left(1 - k^2(1-\tilde{k}^2y^2)^{-1}\right)\right]^{1/2}}. \end{aligned}$$

Some algebra shows

$$\begin{aligned} \frac{1}{1-\tilde{k}^2y^2} - 1 &= \frac{1-(1-\tilde{k}^2y^2)}{1-\tilde{k}^2y^2} = \tilde{k}^2y^2(1-\tilde{k}^2y^2)^{-1} \\ 1 - \frac{k^2}{1-\tilde{k}^2y^2} &= \frac{1-\tilde{k}^2y^2-k^2}{1-\tilde{k}^2y^2} = \frac{\tilde{k}^2-\tilde{k}^2y^2}{1-\tilde{k}^2y^2} = \tilde{k}^2(1-y^2)(1-\tilde{k}^2y^2)^{-1} \end{aligned}$$

and plugging into above, we have

$$\begin{aligned} K'(k) &= \int_0^1 \frac{\tilde{k}^2 y(1-\tilde{k}^2y^2)^{3/2} dy}{\left[\left((1-\tilde{k}^2y^2)^{-1} - 1\right)\left(1 - k^2(1-\tilde{k}^2y^2)^{-1}\right)\right]^{1/2}} \\ &= \int_0^1 \frac{\tilde{k}^2 y(1-\tilde{k}^2y^2)^{-3/2} dy}{\left[\left(\tilde{k}^2y^2(1-\tilde{k}^2y^2)^{-1}\right)\left(\tilde{k}^2(1-y^2)(1-\tilde{k}^2y^2)^{-1}\right)\right]^{1/2}} \\ &= \int_0^1 \frac{(1-\tilde{k}^2y^2)^{-1/2} dy}{(1-y^2)^{1/2}} = K(\tilde{k}) \end{aligned}$$

as required.

(b) Following the hint, let $x = \dfrac{2t}{(1+\tilde{k}+(1-\tilde{k})t^2)}$, then

$$dx = \frac{2(1+\tilde{k}+(1-\tilde{k})t^2) - 4(1-\tilde{k})t^2}{(1+\tilde{k}+(1-\tilde{k})t^2)^2} = \frac{2(1+\tilde{k}-(1-\tilde{k})t^2)}{(1+\tilde{k}+(1-\tilde{k})t^2)^2}$$

and we see that when $x = 0, t = 0$ and when $x = 1$, $t = 1$. Some algebra shows

$$\begin{aligned}
1 - x^2 &= 1 - \left(\frac{2t}{(1 + \tilde{k} + (1 - \tilde{k})t^2)}\right)^2 \\
&= 1 - \frac{4t^2}{(1 + \tilde{k} + (1 - \tilde{k})t^2)^2} \\
&= \frac{(1 + \tilde{k} + (1 - \tilde{k})t^2)^2 - 4t^4}{(1 + \tilde{k} + (1 - \tilde{k})t^2)^2} \\
&= \frac{(1 + \tilde{k})^2 + 2(1 - \tilde{k}^2)t^2 + (1 - \tilde{k})^2 t^4 - 4t^4}{(1 + \tilde{k} + (1 - \tilde{k})t^2)^2} \\
&= \frac{(1 + \tilde{k})^2 - 2(1 + \tilde{k}^2)t^2 + (1 - \tilde{k})^2 t^4}{(1 + \tilde{k} + (1 - \tilde{k})t^2)^2} \\
&= \frac{(1 + \tilde{k})^2}{(1 + \tilde{k} + (1 - \tilde{k})t^2)^2}\left(1 - 2\frac{1 + \tilde{k}^2}{(1 + \tilde{k})^2}t^2 + \frac{(1 - \tilde{k})^2}{(1 + \tilde{k})^2}t^4\right)
\end{aligned}$$

and using the fact that $\tilde{k}^2 = 1 - k^2 \Rightarrow 1 - \tilde{k}^2 = k^2$, we also have

$$\begin{aligned}
1 - k^2x^2 &= 1 - k^2\left(\frac{2t}{(1 + \tilde{k} + (1 - \tilde{k})t^2)}\right)^2 \\
&= 1 - \frac{4k^2t^2}{(1 + \tilde{k} + (1 - \tilde{k})t^2)^2} \\
&= \frac{(1 + \tilde{k} + (1 - \tilde{k})t^2)^2 - 4k^2t^4}{(1 + \tilde{k} + (1 - \tilde{k})t^2)^2} \\
&= \frac{(1 + \tilde{k})^2 + 2(1 - \tilde{k}^2)t^2 + (1 - \tilde{k})^2 t^4 - 4k^2t^4}{(1 + \tilde{k} + (1 - \tilde{k})t^2)^2} \\
&= \frac{(1 + \tilde{k})^2 - 2(1 - \tilde{k}^2)t^2 + (1 - \tilde{k})^2 t^4}{(1 + \tilde{k} + (1 - \tilde{k})t^2)^2} \\
&= \frac{(1 + \tilde{k})^2}{(1 + \tilde{k} + (1 - \tilde{k})t^2)^2}\left(1 - 2\frac{1 - \tilde{k}^2}{(1 + \tilde{k})^2}t^2 + \frac{(1 - \tilde{k})^2}{(1 + \tilde{k})^2}t^4\right) \\
&= \frac{(1 + \tilde{k})^2}{(1 + \tilde{k} + (1 - \tilde{k})t^2)^2}\left(1 - \frac{1 - \tilde{k}}{1 + \tilde{k}}t^2\right)^2
\end{aligned}$$

Plugging into the definition of $K(k)$, we have

$$
\begin{aligned}
&K(k)\\
&= \int_0^1 \frac{dx}{((1-x^2)(1-k^2x^2))^{1/2}}\\
&= \int_0^1 \frac{\frac{2(1+\tilde{k}-(1-\tilde{k})t^2)}{(1+\tilde{k}+(1-\tilde{k})t^2)^2}dt}{\left(\frac{(1+\tilde{k})^2}{(1+\tilde{k}+(1-\tilde{k})t^2)^2}\left(1-2\frac{1+\tilde{k}^2}{(1+\tilde{k})^2}t^2+\frac{(1-\tilde{k})^2}{(1+\tilde{k})^2}t^4\right)\right)^{1/2}\left(\frac{(1+\tilde{k})^2}{(1+\tilde{k}+(1-\tilde{k})t^2)^2}\left(1-\frac{1-\tilde{k}}{1+\tilde{k}}t^2\right)^2\right)^{1/2}}\\
&= \frac{2}{1+\tilde{k}}\int_0^1 \frac{1-\frac{1-\tilde{k}}{1+\tilde{k}}t^2dt}{\left(1-2\frac{1+\tilde{k}^2}{(1+\tilde{k})^2}t^2+\frac{(1-\tilde{k})^2}{(1+\tilde{k})^2}t^4\right)^{1/2}\left(\left(1-\frac{1-\tilde{k}}{1+\tilde{k}}t^2\right)^2\right)^{1/2}}\\
&= \frac{2}{1+\tilde{k}}\int_0^1 \frac{dt}{\left((1-t^2)\left(1-\left(\frac{1-\tilde{k}}{1+\tilde{k}}\right)^2t^2\right)\right)^{1/2}}\\
&= \frac{2}{1+\tilde{k}}K\left(\frac{1-\tilde{k}}{1+\tilde{k}}\right).
\end{aligned}
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References

[SS03] Elias M. Stein and Rami Shakarchi. *Complex Analysis*. Princeton University Press, 2003.